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Some comments on the sensitivity to sensor location of inverse heat conduction problems using Beck's method

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INTRODUCTION

THE INVERSE heat conduction problem (IHCP) is the determination of surface heat flux from transient, measured temperatures inside solids. One of the important parameters involved in IHCP is the dimensionless time step based on the depth of the sensor below the heated surface. Using Beck's method [1], this note presents a comparison of the sensitivity to measurement errors for different sensor locations when the dimensionless time step based on sensor depth is the same. Both cases of identical dimensional and identical dimensionless variances in the temperature measurement are considered.

ANALYSIS

The Beck procedure is to determine at each time step the heat flux that minimizes the least-square error between the calculated and measured temperatures. The least-square function is:

$$F_r = \sum_{j=1}^{r} (T_E^{M+j} - Y^{M+j})^2 \tag{1}$$

where T_E^{M+j} and Y^{M+j} are respectively the calculated and measured temperatures at time $(M+j)\Delta t$. The number of future time temperatures used in the minimization is r. A solution for the unknown heat flux given by Blackwell [2, 3] is:

$$q_E^{M+1} = \sum_{j=1}^{r} K_E^j (Y^{M+j} - \psi_E^{M,j}) \tag{2}$$

with

$$K_E^j = \phi_E^j \Big/ \sum_{j=1}^{r} (\phi_E^j)^2. \tag{3}$$

The symbol $\psi_E^{M,j}$ represents the decay of the temperature at $x = E$ when the heat flux is set to zero for the r future time steps ($q^{M+1} = q^{M+2} = \cdots = q^{M+r} = 0$). ϕ_E^j is the temperature response at $x = E$ of the body initially at zero temperature and subjected to a unit step in heat flux. The ϕs are called sensitivity coefficients and the Ks are called gain coefficients [3]. For a semi-infinite body or for a flat plate at the early stage of the transient, the expression of the sensitivity coefficients [4] is:

$$\phi(x, t) = \frac{1}{k}\{2\sqrt{\alpha t/\pi}\, e^{-x^2/4\alpha t} - x\, \mathrm{erfc}\,(x/2\sqrt{\alpha t})\} \tag{4}$$

which can be written as:

$$\phi(x, t) = x\left\{\frac{1}{k}\left[\frac{2}{\sqrt{\pi}}\sqrt{t^+}\, e^{-1/4t^+} - \mathrm{erfc}\,(1/2\sqrt{t^+})\right]\right\} = x\eta(t^+) \tag{5}$$

where $t^+ = \alpha t/x^2$ is the Fourier number.

The measured temperatures can be considered as the exact temperatures plus random errors:

$$Y^j = \bar{T}^j + \varepsilon^j \tag{6}$$

Substituting equation (6) into (2) gives:

$$q_E^{M+1} = \bar{q}_E^{M+1} + \tilde{q}_E^{M+1} \tag{7a}$$

NOMENCLATURE

- erfc complementary error function
- E depth of thermocouple below heated surface
- E_1 location of the first sensor
- E_2 location of the second sensor
- F_r least-squares function
- K_E^j gain coefficient defined by equation (3)
- k thermal conductivity
- L thickness of the slab
- q^j flux at time $j \cdot \Delta t$
- q_0 nominal value of the heat flux
- $\bar{q}$ heat flux defined by equation (7b)
- $\tilde{q}$ error in the heat flux defined by equation (7c)
- r number of future temperatures
- T temperature
- T_E^j computed temperature at time $j \cdot \Delta t$ and depth E below the heated surface
- $\bar{T}$ exact measured temperature
- t time
- t^+ dimensionless time, $\alpha t/x^2$
- x abscissa
- Y measured temperature.

Greek symbols
- Δt dimensional time step
- Δt_E^+ dimensionless time step, $\alpha \Delta t/E^2$
- ε random error
- α thermal diffusivity
- ϕ_E^j sensitivity coefficient at $x = t$ and time $t_j = j \cdot \Delta t$, equation (5).

Subscripts
- 1 relative to the first sensor
- 2 relative to the second sensor
- L relative to the sensor located at the insulated surface.

Superscripts
- j relative to the time
- $+$ dimensionless value.

where

$$\tilde{q}_E^{M+1} = \sum_{j=1}^{r} K_E^j(\bar{T}^{M+j} - \psi_E^{M,j}) \quad (7b)$$

$$\tilde{q}_E^{M+1} = \sum_{j=1}^{r} K_E^j \varepsilon^{M+j}. \quad (7c)$$

$\bar{q}_E^{M+1}$ is the exact heat flux if $\psi_E^{M,j}$ is exact. It is true for example at the first time step. However, this note focuses on the second term $\tilde{q}_E^{M+1}$ which is the error in the surface heat flux introduced at this particular time step by the random errors $\varepsilon^{M+j}(j = 1, 2, \ldots, r)$. Thus $\tilde{q}_E^{M+1}$ represents the sensitivity to measurement errors. The purpose of this note is to compare this error $\tilde{q}_E^{M+1}$ for different sensor locations when the dimensionless time steps Δt_E^+ based on the sensor depth are equals. Four cases are studied.

The first case corresponds to two sensors located within the solid at E_1 and E_2 ($E_1 < E_2 < L$) and the dimensional variances in the measurement are the same which is usually true for a given experiment. The ratio of the sensitivity coefficients of the two sensors is expressed using equation (5) and $\Delta t_{E_1}^+ = \Delta t_{E_2}^+$ by:

$$\frac{\phi_{E_1}^j}{\phi_{E_2}^j} = \frac{E_1}{E_2} \quad j = 1, 2, \ldots, r. \quad (8)$$

Substituting in equation (3c) gives:

$$\frac{K_{E_1}^j}{K_{E_2}^j} = \frac{E_2}{E_1} \quad j = 1, 2, \ldots, r. \quad (9)$$

The ratio $\tilde{q}_{E_1}^{M+1}/\tilde{q}_{E_2}^{M+1}$ is found using equation (7c) and the substitution of (9) yields:

$$\frac{\tilde{q}_{E_1}^{M+1}}{\tilde{q}_{E_2}^{M+1}} = \frac{E_2}{E_1}. \quad (10)$$

For $E_1 = 0.25$ and $E_2 = 0.5$, equation (10) expresses that the sensitivity to measurement errors is twice that for the sensor closest to the heated surface. It might be surprising but one must keep in mind that in that case the dimensional time step Δt_{E_1} for this sensor is four times smaller than the one at the other sensor.

The second case considered is a finite plate perfectly insulated at the back face $x = L$. This case is often used to study inverse methods and the sensor can be located either within the solid or at the insulated surface. The dimensionless time steps Δt_E^+ are still equal and it will be first assumed that the dimensionless random errors ε^{j+} are identical for the two sensors. This case is of interest for the comparison of inverse methods because the same dimensionless measurement errors can be used in every paper. The dimensionless error is defined by:

$$\varepsilon^{j+} = \frac{k}{q_0 x}\varepsilon^j. \quad (11)$$

So

$$\varepsilon^j = \frac{q_0 x}{k}\varepsilon^{j+}. \quad (12)$$

The ratio $\tilde{q}_E^{M+1}$ over $\tilde{q}_L^{M+1}$ is found using equation (7c) and substituting equation (12) gives:

$$\frac{\tilde{q}_E^{M+1}}{\tilde{q}_L^{M+1}} = \frac{E}{L}\left(\sum_{j=1}^{r} K_E^j \Big/ \sum_{j=1}^{r} K_L^j\right). \quad (13)$$

When $\Delta t_E^+ = \Delta t_L^+$, the dimensionless sensitivity coefficients for the sensor within the solid (ϕ_E^+) and for the sensor at the insulated surface (ϕ_L^+) are related at the early stage of the transient [4] by:

$$\phi_L^{j+} = 2\phi_E^{j+} \quad j = 1, 2, \ldots, r \quad (14)$$

using the definition of ϕ^+:

$$\phi_x^{j+} = \frac{k\phi^j}{x} \quad (15)$$

it gives:

$$\frac{\phi_L^j}{\phi_E^j} = 2\frac{L}{E} \quad j = 1, 2, \ldots, r. \quad (16)$$

Substitution of equations (3) and (16) in (13) yields:

$$\frac{\tilde{q}_E^{M+1}}{\tilde{q}_L^{M+1}} = 2. \quad (17)$$

Thus for the same dimensionless time steps—both based on depth of the sensor—and the same dimensionless random errors in the measurement, the error in the surface heat flux for a sensor at any interior location is twice the error for the sensor at the insulated surface.

In the third case, it is assumed that the dimensional random errors are the same while the second sensor is still at the insulated surface. Thus equation (16) is used instead of equation (8) to get equations similar to (9) and (10), and it yields:

$$\frac{\tilde{q}_E^{M+1}}{\tilde{q}_L^{M+1}} = 2\frac{L}{E}. \quad (18)$$

In the fourth case, the dimensionless random errors are the same and the second sensor is within the solid. Then equation (13) with $E = E_1$ and $L = E_2$ is valid. Substitution of equation

Table 1. Comparison of the sensitivity to measurement errors for different sensor locations for a flat plate insulated at the back face. The dimensionless time steps based on depth of the sensor are identical and are small

Case	Location of the first sensor E_1	Location of the second sensor $E_2(E_2 > E_1)$	Relation between the random error in the measurements	Relation between the error in the heat flux
4	within solid	within solid	same dimensionless errors	$\tilde{q}_{E_1} = \tilde{q}_{E_2}$
1	within solid	within solid	same dimensional errors	$\tilde{q}_{E_1} = \frac{E_2}{E_1}\tilde{q}_{E_2}$
2	within solid	insulated face	same dimensionless errors	$\tilde{q}_{E_1} = 2\tilde{q}_{E_2}$
3	within solid	insulated face	same dimensional errors	$\tilde{q}_{E_1} = 2\frac{E_2}{E_1}\tilde{q}_{E_2}$

(9) gives:

$$\frac{\tilde{q}_{E_1}^{M+1}}{\tilde{q}_{E_2}^{M+1}} = 1. \quad (19)$$

The relations between the errors in the surface heat flux are reported in Table 1 for the four cases.

EXAMPLE

Consider a flat plate insulated on one side. For simplicity the parameters L, α, k and r are set equal to unity. The error in the surface heat flux is compared for three sensors located at $E = 0.25$, $E = 0.5$ and at the back face $L = 1$. Setting $r = 1$ in equations (3) and (7c) gives the expression of the error

$$\tilde{q}_E^{M+1} = K_E^1 \varepsilon^{M+1} = \varepsilon^{M+1}/\phi_E^1. \quad (20)$$

In order to have the same dimensionless time step based on sensor depth, the dimensional time step for the three sensors must vary. The dimensional time steps, the sensitivity coefficients and the errors are reported in Table 2 when $\Delta t_E^+ = 0.16$.

For the two sensors within the solid the error is twice that for the sensor closest to the heated surface, but, it should be noted that the dimensional time step is four times smaller for this sensor. Table 2 also shows that the sensitivity to measurement errors is four times larger at the center of the plate than at the insulated face, but again, the dimensional time step is four times smaller for this sensor.

Table 2. Error in the surface heat flux due to a single measurement error ε^{M+1} for different sensor locations ($\Delta t_E^+ = 0.16$)

Sensor location	Dimensional time step	Sensitivity coefficient ϕ_E^1	Error $\tilde{q}^{M+1}$
Within solid $E = 0.25$	0.01	0.00438	228 ε^{M+1}
Within solid $E = 0.5$	0.04	0.00876	114 ε^{M+1}
Insulated face $E = 1$	0.16	0.03504	28.5 ε^{M+1}

SUMMARY

It has been shown that two inverse heat conduction problems are not always identical despite the dimensionless time steps based on the distance from the heated surface to the sensor are the same. First, the sensitivity to measurement errors is twice for an interior sensor than for a sensor at the insulated surface even if both the dimensionless time steps based on sensor depth and the dimensionless measurement errors are equals. Now, considering an experiment where the dimensional random measurement errors are more likely to be the same for every sensor, the sensitivity to measurement errors is inversely proportional to the sensor depth for a constant dimensionless time step. But it is important to point out that the dimensional time steps are proportional to the square of the sensor depth. Thus the closer the sensor, the smaller the dimensional time step and then the largest the sensitivity to measurement errors. However, the best sensor location for a given dimensional time step and a given dimensional variance in the temperature measurements is near the heated surface. The results presented herein apply to all IHCP algorithms and are of interest for the comparison of IHCP methods. They are only valid when small dimensionless time steps are used which usually is required.

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